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Generalized codes and their application to Ising models with four-spin interactions including the eight-vertex model

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Abstract. Generalized codes are defined for the simple quadratic lattice and it is shown how they may be used to derive high magnetic field or low temperature expansions for two Ising-type models with four-spin interactions. One of these models, namely Baxter's eight-vertex model, has critical exponents which are known to depend on x, the ratio of the strengths of the four-spin to two-spin interactions. The series so derived are analysed to yield estimates of the critical exponent δ for the magnetization as a function of magnetic field at T_c. The results for both models are consistent with a constant value $\delta = 15$ independent of x as predicted by the scaling laws.

1. Introduction

In this paper we introduce generalized codes for the simple quadratic lattice, and show how they may be used to derive low temperature or high magnetic field expansions for several spin $\frac{1}{2}$ Ising-type models on the square lattice. These include, in addition to the standard Ising models with first and with first and second neighbour interactions (models SQ(1) and SQ(1, 2), respectively), two models having four-spin interactions around each of the underlying squares of the lattice. One of these models is the isotropic case of Baxter's eight-vertex model in nonzero magnetic field (Baxter 1972), the other a model first studied by Theodorakopoulos (1972) and independently by Oitmaa and Gibberd (1973). We refer to these as model B and model T/OG, respectively. Finally, we study the exponent δ for both these models by deriving and analysing the first six coefficients (through μ^6) of the series for the magnetization in a magnetic field along the critical isotherm. The results are compared with predictions based upon the scaling laws (Fisher 1967, Barber and Baxter 1973) and the exact, conjectured or estimated values of other exponents.

Other applications of the generalized codes include the derivation of series expansions for the eight-vertex model in nonzero electric and magnetic fields. We remark that the eight-vertex model was originally proposed, not as a magnetic model but as one of ferroelectrics. (The equivalence was first pointed out by Wu (1971) and Kadanoff and Wegner (1971).) The two formulations lead naturally to two types of field, namely magnetic and electric. To deal with this situation, Enting (1973) has introduced a subscript notation: the exponents $\beta, \gamma', \gamma, \delta, \ldots$ (Fisher 1967) are given subscripts e or m depending on whether they refer to electric or magnetic fields, respectively. In this paper, δ_{m} is investigated; however, since no confusion can arise the subscript m is omitted. The corresponding exponent δ_{e} has been studied by Enting and Gaunt (1974).

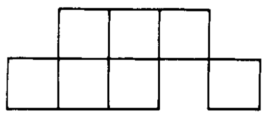
Baxter's exact solution of the eight-vertex model has generated considerable interest in Ising models with multiple spin interactions; for example, Wegner (1971), Baxter and Wu (1973), Wood and Griffiths (1973, 1974) and Griffiths and Wood (1973), which also contains a useful summary of most work except the very recent. It appears that these models may exhibit critical behaviour which is quite different to that of the Ising model with pair interactions only. Using numerical techniques (Gaunt and Guttmann 1974) originally developed for the simple Ising model and similar problems, Griffiths and Wood (1973) predicted that the triangular lattice with pure triplet interactions should have an usually large specific heat exponent α'. This was subsequently confirmed by exact calculation (Baxter and Wu 1973). These techniques have also enabled the dependence of the susceptibility exponent γ (Ditzian 1972) and spontaneous magnetization exponent β (Oitmaa 1974) on the strength of the four-spin interaction to be elucidated for the eight-vertex model. The results are in good agreement with the conjectured variations (Barber and Baxter 1973). The work cited above justifies the application of the conventional techniques of series analysis to the models studied in this paper.

The 'code' method of deriving low temperature or high field expansions for an Ising ferromagnet (or antiferromagnet) has been described in detail elsewhere (Sykes *et al* 1965, 1973a,b,c,d,e to be referred to as I, II, III, IV, V, VI respectively). The general theory is given in I and II.

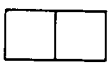
The problem for the simple quadratic lattice may be thought of as one of counting and coding arrangements of squares on the square lattice. This combinatorial problem is closely related to the strong graph expansion (Sykes *et al* 1966) for SQ(1, 2). The general form of a code is

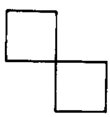
$$(\lambda, \alpha, \beta, \gamma, \delta), \qquad \lambda = \alpha+\beta+\gamma+\delta,$$

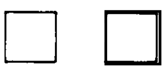
where $\alpha, \beta, \gamma, \delta$ are the number of vertices belonging to 1, 2, 3, 4 squares, respectively. (No confusion should arise between this and the exponent notation.) For example,


$(8N)$

has the code (16, 8, 5, 2, 1). To derive the *complete* nth *code* F_n all the ways in which n squares may be chosen on the simple quadratic lattice must be counted and coded. Thus, for two squares there are three distinct cases:

(*a*)  Edge-to-edge $(2N)$

(*b*)  Corner-to-corner $(2N)$

(*c*)  Separated $(\frac{1}{2}N^2-4\frac{1}{2}N)$

from which we obtain

$$F_2 = 2(6,4,2)+2(7,6,1)-4\tfrac{1}{2}(8,8). \tag{1.1}$$

Sykes *et al* give the first seven complete codes $F_1, F_2, \ldots, F_7$ in I and III. Corresponding results for other lattices have also been derived (see I, III and V).

More generally, for any loose-packed lattice of coordination number q, an nth order code (ie one contained in F_n) may be interpreted by the substitution

$$(\lambda, \alpha, \beta, \gamma, \ldots) = \frac{Y^n(1+bX)^\alpha(1+b^2X)^\beta(1+b^3X)^\gamma\ldots}{(1+X)^\lambda}, \qquad \lambda = \alpha+\beta+\gamma+\ldots, \tag{1.2}$$

where the length of the integer sequence $\lambda, \alpha, \beta, \gamma, \ldots$ never exceeds $(q+1)$. After expansion of the right-hand side of (1.2) the coefficient of $X^sY^nb^r$ represents the contribution of s overturned spins on one sublattice (by convention, the A sublattice), n overturned spins on the other sublattice (B spins) having r nearest neighbour links between them. A second substitution

$$X = Y = \mu u^{q/2}, \qquad b = u^{-1}, \tag{1.3}$$

is required to make contact with the low temperature Ising variables

$$\mu = \exp\left(\frac{-2mH}{kT}\right), \qquad u = \exp\left(\frac{-4J}{kT}\right). \tag{1.4}$$

(The usual notation is employed—see I and II, for example.) The precise way in which the first N complete codes determine the first $(2N+1)$ L-polynomials in the high-field expansion of the configurational free energy,

$$\ln \Lambda = \sum_{s=1}^{\infty} L_s(u)\mu^s \tag{1.5}$$

is described in I and II. The high-field polynomials are given in I, III and V.

If (1.5) is re-arranged as a u grouping or low temperature expansion

$$\ln \Lambda = \sum_s \psi_s(\mu)u^s, \tag{1.6}$$

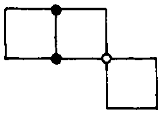
relatively few complete ψ polynomials are obtained; this is because quite low powers of u can come from high-order L polynomials. To extend the u grouping, the leading terms in higher L polynomials must be determined. These extra coefficients can be obtained by enumerating only those codes whose expansions will make a contribution to the required coefficient. This leads to the concept of a *partial code* $F_n^{(m)}$, which contains the subset of codes of F_n required in the derivation of the u grouping correct through ψ_m. Further details are given in IV and VI.

In the next section, the simple quadratic code system outlined above is generalized so as to contain additional detailed information about the underlying graphs. The first six *generalized complete codes* are derived. Their interpretation is quite different to that described by the substitutions (1.2) and (1.3) above. Instead, direct substitutions are developed (see equations (2.13), (2.14), (2.17), (2.18) and (2.19)) which are more akin to the substitutions used by Sykes *et al* in II for interpreting the honeycomb and diamond codes on the triangular and face-centred cubic lattices, respectively. These substitutions enable us to derive the first six terms of a high-field expansion—the analogue of (1.5)—for various Ising models, including SQ(1, 2), B and T/OG. The low temperature zero-field series derived by Oitmaa (1974) for models B and T/OG are then checked by deriving *generalized partial codes*; these also contain the complete field dependence.

2. Generalized codes: their definition and some applications

2.1. Generalized complete codes

We begin by observing that the mutual contact of two squares at a common vertex can occur in two quite distinct ways depending on whether the squares in question do or do not touch along an edge. The situation is easily understood with the aid of a simple example,



in which the two types of second-order contact are distinguished by full (type 1) and open (type 2) circles. If β_1 and β_2 are the number of second-order contacts of type 1 and type 2, respectively, then clearly

$$\beta_1+\beta_2 = \beta. \tag{2.1}$$

We may now define a generalized code; the general form is

$$(\lambda, \alpha, \beta_1|\beta_2, \gamma, \delta), \qquad \lambda = \alpha+\beta_1+\beta_2+\gamma+\delta. \tag{2.2}$$

Thus, a generalized code is nothing more than a code in which the information about the two types of second-order contact is kept separate. As an example, the configuration of seven squares drawn in § 1, having the 7th order code (16, 8, 5, 2, 1), has the generalized 7th order code (16, 8, 4|1, 2, 1).

Note that there is no corresponding subdivision for α, γ and δ; this is because first-order, third-order and fourth-order contacts are of one type only.

The analogue of F_n is the *generalized complete* nth *code*

$$G_n = \sum{}' (\lambda, \alpha, \beta_1|\beta_2, \gamma, \delta) \tag{2.3}$$

where here and below Σ' indicates a sum over all generalized nth order codes. (As usual, we shall not introduce a notation for the occurrence factors.) For example, from the three possibilities for two squares drawn in § 1, we get

$$G_2 = 2(6, 4, 2|0)+2(7, 6, 0|1)-4\tfrac{1}{2}(8, 8) \tag{2.4}$$

in place of (1.1). Likewise the first six generalized complete codes G_1 through G_6 have been derived and are given in appendix 2. We have checked that these data reduce correctly to the F_n when the distinction between the two types of second-order contact is not retained, ie

$$F_n = \sum{}' (\lambda, \alpha, \beta_1+\beta_2, \gamma, \delta). \tag{2.5}$$

2.2. Applications

Our main application is the derivation of high magnetic field expansions for two models with the hamiltonian

$$\mathscr{H} = -J \sum_{\langle ij\rangle} \sigma_i\sigma_j - J_4 \sum_{\langle ijkl\rangle} \sigma_i\sigma_j\sigma_k\sigma_l - mH \sum_i \sigma_i \tag{2.6}$$

where the σ variables have the values ± 1. The four-spin interaction is of strength J_4 and the appropriate summation is over all basic squares of the simple quadratic lattice. The pair interaction of strength J is summed over either nearest neighbour sites or next nearest neighbour sites corresponding to models T/OG and B, respectively. The last summation in (2.6) is taken over all lattice sites.

The free energy per spin is

$$f = -2J - J_4 - mH - kT \ln \Lambda, \tag{2.7}$$

where power series expansions for the configurational free energy $\ln \Lambda$ can be derived by enumerating successive perturbations from the ordered ground state. Oitmaa (1974) employed a technique used by Fan and Wu (1969) to show how this may be conveniently done by enumerating 'polygon' configurations on the dual lattice and associating weight factors with the different vertex types. He finds

$$\ln \Lambda = \sum_{\{G\}} C_G u^a w^b \mu^c, \tag{2.8}$$

where

$$u = \exp\left(\frac{-4J}{kT}\right), \qquad w = \exp\left(\frac{-2J_4}{kT}\right), \qquad \mu = \exp\left(\frac{-2mH}{kT}\right). \tag{2.9}$$

The sum is over all graphs G embeddable in the simple quadratic lattice having vertices of even order ('polygons') and C_G is the weak lattice constant of G (Sykes *et al* 1966). Expressions for a, b and c are:

Model T/OG

$$\begin{aligned} a &= \tfrac{1}{2} n_{\mathrm{B}} \\ b &= n_{2\mathrm{E}} \\ c &= n_0 + \tfrac{1}{2} n_{\mathrm{B}} - n_4 - n_{\mathrm{C}} + n_{\mathrm{H}} \end{aligned} \tag{2.10}$$

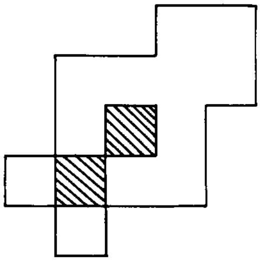
Model B

$$\begin{aligned} a &= n_{\mathrm{B}} - 2n_4 - \tfrac{1}{2} n_{2\mathrm{E}} \\ b &= n_{2\mathrm{E}} \\ c &= n_0 + \tfrac{1}{2} n_{\mathrm{B}} - n_4 - n_{\mathrm{C}} + n_{\mathrm{H}} \end{aligned} \tag{2.11}$$

where n_{B}, $n_{2\mathrm{E}}$, n_4, n_0, n_{C} and n_{H} are the number of bonds, elbows, fourth-order vertices, unoccupied interior sites, components and holes of G, respectively. Elbows are second-order vertices which are the meet of perpendicular bonds. It is often necessary to distinguish between the two types of elbow that can occur and this may be conveniently done by assigning a direction to G such that it is traversed in a clockwise direction with respect to its interior. Denoting the number of right-hand and left-hand elbows by $n_{2\mathrm{R}}$ and $n_{2\mathrm{L}}$, respectively, we have

$$n_{2\mathrm{E}} = n_{2\mathrm{R}} + n_{2\mathrm{L}}. \tag{2.12}$$

The meaning of the other terms (bonds, fourth-order vertices, holes, etc) should be self-evident but further clarification may be provided by an example:



This has $n_{\mathrm{B}} = 28$, $n_{2\mathrm{E}} = 14$ ($n_{2\mathrm{R}} = 9$, $n_{2\mathrm{L}} = 5$), $n_4 = 4$, $n_0 = 1$, $n_{\mathrm{C}} = 1$ and $n_{\mathrm{H}} = 2$. For further details, see Oitmaa (1974). Note, however, that his expression for c is identical to that in (2.10) and (2.11) but with $n_{\mathrm{H}} \equiv 0$; it appears therefore that Oitmaa has overlooked the possibility of holes unless n_{H} is somehow absorbed into his definition of n_0.

To relate these models to the generalized codes, we first notice that there is a one-to-one correspondence between the polygonal graphs contributing to (2.8) and the possible arrangements of squares on the square lattice. Furthermore, it can be shown that a, b and c for a particular polygonal graph are determined by the parameters α, β_1, β_2, γ and the order n of the generalized code for the corresponding arrangement of squares. Thus, we find (see appendix 1 for proof)

$$\ln \Lambda = \sum{}' u^{\frac{1}{2}(\alpha+\beta_1+2\beta_2+\gamma)} w^{\alpha+\gamma} \mu^n \qquad \text{(model T/OG)} \tag{2.13}$$

and

$$\ln \Lambda = \sum{}' u^{\frac{1}{2}(\alpha+2\beta_1+\gamma)} w^{\alpha+\gamma} \mu^n, \qquad \text{(model B)} \tag{2.14}$$

where the occurrence factors have been omitted in accord with our convention. The weak lattice constants C_G appearing in (2.8) are obtained by collecting together the occurrence factors of those terms in (2.13) or (2.14) with the same powers of u, w, μ. Using G_1 through G_6 we now obtain the high-field expansions

$$\ln \Lambda = \sum_{s=1}^{\infty} L_s(u, w)\mu^s \tag{2.15}$$

for models T/OG and B correct through μ^6. These are not given here explicitly since the substitutions (2.13) and (2.14) are quite straightforward.

The generalized complete codes may also be used to derive high-field expansions

$$\ln \Lambda = \sum_{s=1}^{\infty} g_s(u, v)\mu^s \tag{2.16}$$

for model SQ(1, 2): the standard Ising model with first and second neighbour interactions. The required substitution is clearly

$$\ln \Lambda = \sum{}' u^{\frac{1}{2}(\alpha+\beta_1+2\beta_2+\gamma)} v^{\frac{1}{2}(\alpha+2\beta_1+\gamma)} \mu^n, \qquad \text{(model SQ(1, 2))}, \tag{2.17}$$

where the powers of u and v are simply the powers of u in (2.13) and (2.14), respectively. The high-field polynomials g_1 through g_6 derived in this way are in precise agreement with the results of Dalton and Wood (1969); this provides a further stringent check on the correctness of G_1 through G_6.

If $v = u^{1/2}$, ie the nearest neighbour interaction is twice the next nearest neighbour interaction, then (2.17) becomes

$$\ln \Lambda = \sum{}' u^{\frac{3}{4}\alpha + (\beta_1 + \beta_2) + \frac{3}{4}\gamma} \mu^n.$$

Since β_1, β_2 occur only in the combination $(\beta_1 + \beta_2)$, the generalized codes are not really necessary for the derivation of this expansion; the conventional codes $(\lambda, \alpha, \beta, \gamma, \delta)$ contain sufficient information for this purpose as was noted in II, equation (3.10), (3.11) and associated discussion.

By setting $w = 1$ in (2.13) and (2.14), we obtain the simple Ising model with nearest neighbour interactions only,

$$\ln \Lambda = \sum{}' {}^{\frac{1}{2}(\alpha + \beta_1 + 2\beta_2 + \gamma)} \mu^n, \qquad \text{(model SQ(1))} \tag{2.18}$$

or

$$\ln \Lambda = \sum{}' u^{\frac{1}{2}(\alpha + 2\beta_1 + \gamma)} \mu^n, \qquad \text{(model SQ(1))}. \tag{2.19}$$

Alternatively, these results may be obtained by setting $v = 1$ or $u = 1$, respectively, in (2.17). Direct substitution in (2.18) or (2.19) using G_1 through G_6 recaptures the first six polynomials L_1 through L_6 in (1.5). This method of derivation is quite different to that described in § 1 of writing down the F_n (using the known G_n for example) and interpreting the individual codes by the substitutions (1.2) and (1.3). As mentioned previously the direct substitutions (2.13), (2.14), (2.17), (2.18) and (2.19) have more in common with the substitutions developed in II for interpreting the honeycomb and diamond codes on the triangular and face-centred cubic lattices, respectively.

2.3. Generalized partial codes and low temperature expansions

Low temperature expansions analogous to (1.6) may be obtained on grouping (2.13) or (2.14) as

$$\ln \Lambda = \sum_{r,s} \psi_{r,s}(\mu) u^r w^s. \tag{2.20}$$

To obtain a useful number of terms we require the contributions from generalized codes of order $n > 6$. Let us define the *generalized partial code* $G_n^{(m)}$ as containing those generalized codes necessary for the derivation of the $\psi_{rs}(\mu)$—polynomials for all values of s $(\leqslant 2r)$ when $r = 2, 3, \ldots, m$. We have used our unpublished data to deduce the $G_n^{(m)}$ through $m = 8$ (model T/OG) and $m = 7$ (model B). These data, which are given in appendix 3, enable us to check the low temperature zero-field expansions given by Oitmaa (1974). He expands the spontaneous magnetization in the form

$$M_0 = 1 - 2\mu \frac{\partial}{\partial \mu} \ln \Lambda \bigg|_{\mu = 1} = 1 - 2 \sum_r h_r(w) u^r \tag{2.21}$$

where $h_r(w)$ are polynomials in w of degree less than or equal to $2r$. (In Oitmaa's notation the $h_r(w)$ polynomials are denoted by $L_r(w)$.) Using appendix 3 we are able to calculate the polynomials complete through $h_m(w)$ where $m = 8$ (model T/OG) and $m = 7$ (model B). The results are in precise agreement with those given by Oitmaa.

In addition, Oitmaa gives the leading terms $u^r w^s$ in several higher-order polynomials correct through $r+s = 15$ (model T/OG) and $r+s = 18$ (model B). To check these, the contributions from additional generalized codes are also required, and these are given in appendix 4. Again we obtain complete agreement with Oitmaa's results, except for the coefficient of w^6 in $h_9(w)$ for model T/OG. We find

$$\psi_{9,6}(\mu) = 40\mu^7+32\mu^8+24\mu^9+40\mu^{10}+32\mu^{11}+24\mu^{12}+24\mu^{13}+32\mu^{14} +8\mu^{15}+32\mu^{16}+24\mu^{17}+16\mu^{18}+8\mu^{19} \tag{2.22}$$

which yields $4032w^6$ for the required term; Oitmaa gives $3976w^6$†. We believe that Oitmaa has overlooked a configuration contributing to (2.22), possibly an $8\mu^7$. Needless to say, such a small discrepancy would make no practical difference to his estimates of the exponent β.

We emphasize that although the main application we have made of the generalized codes in appendices 3 and 4 is to check Oitmaa's zero-field series for M_0, they contain enough information to derive the complete field dependence of the low temperature expansions. Indeed we have derived and analysed (unpublished) the low temperature series for the zero-field susceptibility of Baxter's model. Unfortunately, convergence is too slow for us to draw any conclusions about the exponent γ'.

3. Estimation of the critical exponent δ

In the previous section we derived the high-field expansion

$$\ln \Lambda = \sum_{s=1}^{\infty} L_s(u, w)\mu^s \tag{3.1}$$

for models T/OG and B, correct through μ^6. The magnetization along the critical isotherm ($T = T_c$) is given by

$$M(x;\mu) = 1-2\sum_{s=1}^{\infty} L_s(u_c, w_c)\mu^s, \tag{3.2}$$

where u_c, w_c are the values of u, w at T_c for some value of

$$x = J_4/J. \tag{3.3}$$

For model T/OG, estimates of T_c derived from high temperature susceptibility series are given by Oitmaa and Gibberd (1973) for several values of x; for model B, T_c may be calculated exactly for any x from Baxter's solution (Baxter 1972). In this last section we analyse the first six terms of the series (3.2) for

$$x = -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2, 3, 4, 5,$$

and estimate the exponent δ defined by

$$M(x;\mu) \backsim (1-\mu)^{1/\delta}, \qquad (T = T_c). \tag{3.4}$$

The possible variation of δ with x is investigated and the results compared with predictions obtained from the scaling laws. Unfortunately, as the series are not very long, the results tend to be somewhat inconclusive and should only be regarded as tentative. For comparison, the accepted result $\delta = 15\pm\frac{1}{2}\%$ for the conventional Ising model is based upon the first fifteen terms for the simple quadratic lattice (Gaunt and Sykes 1972). Although reasonable estimates may be obtained from only six terms, the confidence limits are correspondingly larger (see figure 1 or 2 at $x = 0$).

† Dr Oitmaa has informed the author that he now concurs with this value.

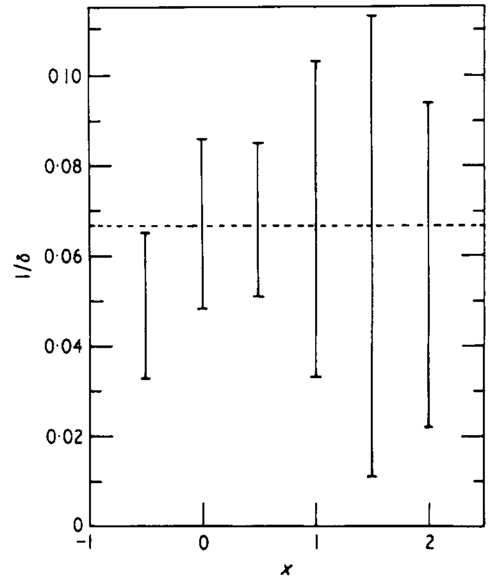


Figure 1. Padé estimates of $1/\delta$ plotted against x for model T/OG. The broken line represents $\delta = 15$ independent of x.

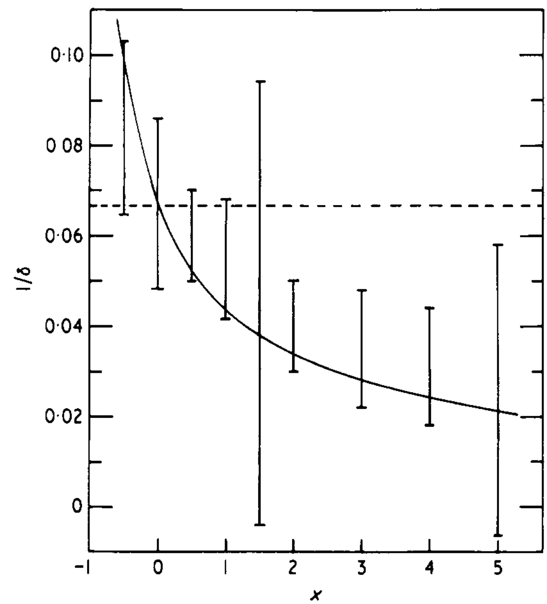


Figure 2. Padé estimates of $1/\delta$ plotted against x for model B. The broken line represents $\delta = 15$ independent of x. The full curve is $\delta = (7+a)/(1-a)$, which although providing a good fit, is quite wrong.

All the standard techniques of series analysis as reviewed by Gaunt and Guttmann (1974) have been employed. The best results are those obtained by evaluating Padé approximants to the $(1-\mu)(\mathrm{d}/\mathrm{d}\mu)\ln M$ series at $\mu = 1$. These estimates of $1/\delta$ are plotted against x in figures 1 and 2.

For model T/OG, the estimates were too irregular to draw any conclusions for $x > 2$. Oitmaa (1974) found an identical situation in his analysis of β. The remaining results are consistent with the hypothesis that $\delta = 15$ for all values of x. ($\delta = 15$ is the accepted result for $x = 0$ (Gaunt and Sykes 1973).) Admittedly the estimate for $x = -\frac{1}{2}$ is rather low but it should be remembered that as usual the error bars are not strict bounds but merely consistency limits (Gaunt and Guttmann 1974). (This is particularly clear in figure 2, where the error bars should not be interpreted as implying the accuracy for $x = 1$ and $x = 2$ is more than four times greater than it is for $x = 1\frac{1}{2}$.) Oitmaa (1974) was also unable to draw any conclusions for $x = -\frac{1}{2}$ due to poor convergence. It is of course possible to interpret results with such large uncertainties in terms of a continuous variation with x. However, the expectation that $\beta = \frac{1}{8}$ and $\gamma = 1\frac{3}{4}$ independent of x (Oitmaa 1974, Theodorakopoulos 1972, Oitmaa and Gibberd 1973) suggests via the scaling laws that $\delta = 15$ independent of x is more likely.

For the eight-vertex model (model B), it is not possible to fit the estimates in figure 2 by a constant value of δ for all $-\frac{1}{2} \leqslant x \leqslant 5$. A continuous variation with x, on the other hand, is readily accommodated; indeed, if, as appears likely, β and α are both linear functions of a^{-1} (Baxter 1972, Barber and Baxter 1973) where

$$a = \frac{\bar{\mu}}{\pi} = 1 - \frac{1}{\pi}\cos^{-1}(\tanh 2J_4/kT_c), \tag{3.5}$$

then in general we should expect δ to be of the form

$$\delta = \frac{A+Ba}{1-Ca}. \tag{3.6}$$

(We mention in passing that Enting and Gaunt (1974) found series estimates of δ_e to be well described by (3.6) with $A = 3$, $B = C = 1$.) The curve in figure 2 corresponds to the simple choice

$$A = 7, \qquad B = C = 1. \tag{3.7}$$

However, since this curve has nonzero gradient at $x = 0$ it disagrees with the first-order perturbation theory of Kadanoff and Wegner (1971). Even worse, if we use the exact result for $\alpha' = \alpha$ (Baxter 1972) and the highly plausible conjecture for β (Barber and Baxter 1973), the curve violates the rigorous thermodynamic inequality $\alpha' + \beta(1+\delta) \geqslant 2$ for $x < 0$ (Griffiths 1965). Although it is just possible to construct curves through the error bars in figure 2 which are consistent with Griffiths inequality and have $\delta = 15$ with nonzero slope at $x = 0$, none of these curves have the form (3.6).

According to the scaling laws, the results for α and β (Baxter 1972, Barber and Baxter 1973) imply $\delta = 15$ independent of x which corresponds to $A = 15$, $B = C = 0$ in (3.6). Since a breakdown of three-exponent scaling is rather unlikely—indeed it is known to be satisfied by the Ising case of the eight-vertex model and by the modified F-model case (Brascamp *et al* 1973)—we seek an alternative explanation of the results in figure 2. We note first that in the interval $-\frac{1}{2} \leqslant x \leqslant 1\frac{1}{2}$, the estimates can be fitted by a constant value of $\delta = 15$. If this value pertains for all x, the misleading nature of the estimates in figure 2 for $x \geqslant 2$ must presumably be attributed to slow convergence. In this connection, Ditzian (1972) found no convergence for $x \gtrsim 1{\cdot}3$ in her analysis of γ, while Oitmaa's (1974) analysis of β was restricted to $x \leqslant 3$ for lack of convergence. Even in this region Barber and Baxter's conjectured β lies outside (and above) the error bars given by Oitmaa for $x > 0$ in figure 3 of his paper; the farther away from $x = 0$, the greater the discrepancy that was found. We conclude therefore that there is evidence to suggest a quite rapid deterioration in convergence as x increases. This conclusion receives further support from an analysis of the μ series for $-\mu(\mathrm{d}/\mathrm{d}\mu)\ln M$, the coefficients of which must approach $1/\delta$. For $x \geqslant \frac{1}{2}$ the coefficients exhibit an odd/even oscillation which increases with increasing x and makes extrapolation rather difficult. The alternation is due to a non-physical singularity on the negative real μ axis moving closer to the origin as x increases. (A similar effect is also found for model T/OG.) We transform therefore to a new variable defined by

$$y = \frac{2\mu}{1+D\mu},$$

where D is chosen (by Padé analysis of the $(\mathrm{d}/\mathrm{d}\mu)\ln M\,(x;\mu)$ series) so as to transform the non-physical singularity far from the origin. The nth coefficient m_n of the y series for $-\mu(\mathrm{d}/\mathrm{d}\mu)\ln M$ must now approach $1/2\delta y_{\mathrm{c}}^{n-1}$ as $n \to \infty$. Plots of $2m_n y_{\mathrm{c}}^{n-1}$ against $1/n$ for $n \leqslant 6$ are smoothly varying, concave upward and have a degree of curvature which increases with increasing x. Assuming this behaviour continues for large n, we obtain, by extrapolating the last pair of points for each value of x, a lower bound for $1/\delta$ which turns out to lie close to the centre of the corresponding error bar in figure 2. Hence, the estimates in figure 2 are underestimates of the true value of $1/\delta$ for $x \geqslant \frac{1}{2}$, the discrepancy increasing with increasing x. We conclude that figure 2 is not inconsistent with a constant value of $\delta = 15$ independent of x.

To summarize, we have found that series estimates of δ for Baxter's eight-vertex model (model B), and a model introduced by Theodorakopoulos and independently by Oitmaa and Gibberd (model T/OG), are not sufficiently precise for us to draw any firm conclusions. However, in the ranges $-\frac{1}{2} \leqslant x \leqslant 1\frac{1}{2}$ (model B) and $-\frac{1}{2} \leqslant x \leqslant 2$ (model T/OG), they are consistent with the scaling prediction that $\delta = 15$ independent of $x(= J_4/J)$.

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Appendix 1. Proof of (2.13) and (2.14)

We must prove

$$a = \tfrac{1}{2}(\alpha+\beta_1+2\beta_2+\gamma), \qquad b = \alpha+\gamma, \qquad c = n \tag{A.1}$$

for model T/OG, and

$$a = \tfrac{1}{2}(\alpha+2\beta_1+\gamma), \qquad b = \alpha+\gamma, \qquad c = n \tag{A.2}$$

for model B, where a, b, c are given by (2.10) and (2.11).

Clearly

$$n_{2\mathrm{R}} = \alpha, \qquad n_{2\mathrm{L}} = \gamma \tag{A.3}$$

and hence

$$b = n_{2E} = n_{2R} + n_{2L} = \alpha + \gamma \tag{A.4}$$

for both models.

By considering the number of bonds emanating from each type of vertex of a polygonal graph, we get

$$n_B = \alpha + \beta_1 + 2\beta_2 + \gamma. \tag{A.5}$$

Hence, for model T/OG

$$a = \tfrac{1}{2}n_B = \tfrac{1}{2}(\alpha + \beta_1 + 2\beta_2 + \gamma), \tag{A.6}$$

while for model B

$$a = n_B - 2n_4 - \tfrac{1}{2}n_{2E} = \tfrac{1}{2}(\alpha + 2\beta_1 + \gamma) \tag{A.7}$$

where we have substituted for n_{2E} and n_B from (A.4) and (A.5) respectively, and have used

$$n_4 = \beta_2. \tag{A.8}$$

If we complete all internal bonds of a polygonal graph to obtain the corresponding arrangement of n squares, Euler's law gives

$$(n + n_H) = (n_B + n_I) - n_S + n_C. \tag{A.9}$$

Here n_I and n_S are the number of internal bonds and total number of sites respectively, and are given by

$$n_I = 2n - \tfrac{1}{2}n_B \tag{A.10}$$

$$n_S = \alpha + \beta_1 + \beta_2 + \gamma + \delta. \tag{A.11}$$

Substitution for n_C using (A.9) and then for n_I using (A.10) gives

$$c = n_0 + \tfrac{1}{2}n_B - n_4 - n_C + n_H = n + n_B - n_4 - n_S + n_0 \tag{A.12}$$

for both models. Finally, substituting for n_B, n_4, n_S, n_0 in (A.12) using (A.5), (A.8), (A.11) and

$$n_0 = \delta, \tag{A.13}$$

we get $c = n$.

We note that the above equations are easily related to equations (3) in Oitmaa's paper. Indeed the first of his equations is identical to (A.5) above when written in his notation. His second equation is obtained from (A.9) by first substituting for n_I from (A.10), and then for n_B, n_S and n using (A.5), (A.11) and

$$4n = \alpha + 2(\beta_1 + \beta_2) + 3\gamma + 4\delta. \tag{A.14}$$

This gives

$$\alpha - \gamma = 2\beta_2 + 4n_C - 4n_H \tag{A.15}$$

or in his notation (using (A.3) and (A.8))

$$n_{2R} - n_{2L} = 2n_4 + 4n_C - 4n_H. \tag{A.16}$$

Again his expression has $n_H \equiv 0$.

Appendix 2. Generalized complete codes G_n

$$G_1 = 1(4,4)$$

$$G_2 = 2(6,4,2|0)+2(7,6,0|1)-4\tfrac{1}{2}(8,8)$$

$$G_3 = 2(8,4,4|0)+4(8,5,2|0,1)+8(9,6,2|1)-24(10,8,2|0)$$
$$+6(10,8,0|2)-28(11,10,0|1)+32\tfrac{1}{3}(12,12)$$

$$G_4 = 1(9,4,4|0,0,1)+2(10,4,6|0)+8(10,5,4|0,1)+8(10,6,2|0,2)$$
$$+16(11,6,4|1)+20(11,7,2|1,1)+1(12,8,0|4)+36(12,8,2|2)$$
$$-61(12,8,4|0)-60(12,9,2|0,1)+18(13,10,0|3)-208(13,10,2|1)$$
$$-150(14,12,0|2)+290(14,12,2|0)+362(15,14,0|1)-283\tfrac{1}{4}(16,16)$$

$$G_5 = 8(11,5,4|0,1,1)+2(12,4,8|0)+12(12,5,6|0,1)+28(12,6,4|0,2)$$
$$+4(12,6,4|1,0,1)+12(12,7,2|0,3)+1(12,8,0|0,4)+24(13,6,6|1)$$
$$+4(13,7,2|3,1)+80(13,7,4|1,1)+48(13,8,2|1,2)-16(13,8,4|0,0,1)$$
$$+4(14,8,2|4)+108(14,8,4|2)-112(14,8,6|0)+88(14,9,2|2,1)$$
$$-296(14,9,4|0,1)-144(14,10,2|0,2)+8(15,10,0|5)+152(15,10,2|3)$$
$$-752(15,10,4|1)-564(15,11,2|1,1)+34(16,12,0|4)-1432(16,12,2|2)$$
$$+1238(16,12,4|0)+804(16,13,2|0,1)-712(17,14,0|3)+4008(17,14,2|1)$$
$$+2866(18,16,0|2)-3604(18,16,2|0)-4672(19,18,0|1)+2771\tfrac{1}{5}(20,20)$$

$$G_6 = 2(12,4,6|0,0,2)+8(13,5,6|0,1,1)+28(13,6,4|0,2,1)+4(13,7,2|0,3,1)$$
$$+2(14,4,10|0)+16(14,5,8|0,1)+6(14,6,4|2,2)+60(14,6,6|0,2)$$
$$+8(14,6,6|1,0,1)+72(14,7,4|0,3)+40(14,7,4|1,1,1)$$
$$+24(14,8,2|0,4)+32(15,6,8|1)+8(15,7,4|3,1)+180(15,7,6|1,1)$$
$$+16(15,8,2|3,2)+302(15,8,4|1,2)+18(15,8,4|2,0,1)$$
$$-40(15,8,6|0,0,1)+84(15,9,2|1,3)-152(15,9,4|0,1,1)$$
$$+8(15,10,0|1,4)+12(16,8,4|4)+228(16,8,6|2)-178(16,8,8|0)$$
$$+36(16,9,2|4,1)+552(16,9,4|2,1)-796(16,9,6|0,1)+216(16,10,2|2,2)$$
$$-1126(16,10,4|0,2)-130(16,10,4|1,0,1)-252(16,11,2|0,3)$$
$$-21(16,12,0|0,4)+2(17,10,0|7)+48(17,10,2|5)+620(17,10,4|3)$$
$$-1872(17,10,6|1)+284(17,11,2|3,1)-4016(17,11,4|1,1)$$
$$-1472(17,12,2|1,2)+222(17,12,4|0,0,1)+40(18,12,0|6)$$
$$+466(18,12,2|4)-6696(18,12,4|2)+3572\tfrac{2}{3}(18,12,6|0)$$
$$-3852(18,13,2|2,1)+6632(18,13,4|0,1)+2128(18,14,2|0,2)$$
$$-82(19,14,0|5)-8480(19,14,2|3)+21\,476(19,14,4|1)$$

$$+11\,500(19,15,2|1,1)-2818(20,16,0|4)+37\,880(20,16,2|2)$$
$$-21\,920(20,16,4|0)-10\,604(20,17,2|0,1)+18\,768\tfrac{2}{3}(21,18,0|3)$$
$$-69\,224(21,18,2|1)-49\,464(22,20,0|2)+45\,830(22,20,2|0)$$
$$+60\,860(23,22,0|1)-29\,096\tfrac{1}{2}(24,24)$$

Appendix 3. Generalized partial codes $G_n^{(m)}$

Model T/OG

$$\begin{aligned}G_7^{(8)} = {} & 16(14,5,6|0,1,2)+6(14,6,4|0,2,2)+4(15,5,6|1,3)\\ & +8(15,5,8|0,1,1)+64(15,6,6|0,2,1)+8(15,6,6|1,0,2)\\ & +80(15,7,4|0,3,1)+16(15,8,2|0,4,1)+2(16,4,12|0)\\ & +20(16,5,10|0,1)+104(16,6,8|0,2)+8(16,6,8|1,0,1)\\ & +216(16,7,6|0,3)+140(16,7,6|1,1,1)+184(16,8,4|0,4)\\ & +168(16,8,4|1,2,1)-40(16,8,6|0,0,2)+40(16,9,2|0,5)\\ & +28(16,9,2|1,3,1)-48(17,8,8|0,0,1)-648(17,9,6|0,1,1)\\ & -616(17,10,4|0,2,1)-88(17,11,2|0,3,1)\end{aligned}$$

$$\begin{aligned}G_8^{(8)} = {} & 2(15,4,8|0,0,3)+4(15,5,6|0,1,3)+1(16,4,8|0,4)\\ & +16(16,5,8|0,1,2)+68(16,6,6|0,2,2)+48(16,7,4|0,3,2)\\ & +2(16,8,2|0,4,2)+8(17,5,10|0,1,1)+100(17,6,8|0,2,1)\\ & +18(17,6,8|1,0,2)+300(17,7,6|0,3,1)+80(17,7,6|1,1,2)\\ & +238(17,8,4|0,4,1)+36(17,8,4|1,2,2)+28(17,9,2|0,5,1)\\ & +2(17,10,0|0,6,1)-110\tfrac{1}{2}(18,8,8|0,0,2)-368(18,9,6|0,1,2)\\ & -138(18,10,4|0,2,2)\end{aligned}$$

$$\begin{aligned}G_9^{(8)} = {} & 1(16,4,8|0,0,4)+16(17,5,8|0,1,3)+48(17,6,6|0,2,3)\\ & +8(17,7,4|0,3,3)+16(18,5,10|0,1,2)+148(18,6,8|0,2,2)\\ & +8(18,6,8|1,0,3)+300(18,7,6|0,3,2)+20(18,7,6|1,1,3)\\ & +172(18,8,4|0,4,2)+20(18,9,2|0,5,2)-48(19,8,8|0,0,3)\\ & -96(19,9,6|0,1,3)\end{aligned}$$

$$\begin{aligned}G_{10}^{(8)} = {} & 2(18,4,10|0,0,4)+16(18,5,8|0,1,4)+12(18,6,6|0,2,4)\\ & +16(19,5,10|0,1,3)+116(19,6,8|0,2,3)+4(19,6,8|1,0,4)\\ & +268(19,7,6|0,3,3)+78(19,8,4|0,4,3)+4(19,9,2|0,5,3)\\ & -25(20,8,8|0,0,4)\end{aligned}$$

$$G_{11}^{(8)} = 8(19, 5, 8|0, 1, 5) + 24(20, 5, 10|0, 1, 4) + 136(20, 6, 8|0, 2, 4) + 128(20, 7, 6|0, 3, 4) + 22(20, 8, 4|0, 4, 4)$$

$$G_{12}^{(8)} = 2(20, 4, 10|0, 0, 6) + 2(21, 4, 12|0, 0, 5) + 12(21, 5, 10|0, 1, 5) + 92(21, 6, 8|0, 2, 5) + 44(21, 7, 6|0, 3, 5) + 1(21, 8, 4|0, 4, 5)$$

$$G_{13}^{(8)} = 24(22, 5, 10|0, 1, 6) + 40(22, 6, 8|0, 2, 6) + 4(22, 7, 6|0, 3, 6)$$

$$G_{14}^{(8)} = 16(23, 5, 10|0, 1, 7) + 6(23, 6, 8|0, 2, 7)$$

$$G_{15}^{(8)} = 2(24, 4, 12|0, 0, 8) + 4(24, 5, 10|0, 1, 8)$$

$$G_{16}^{(8)} = 1(25, 4, 12|0, 0, 9)$$

Model B

$$G_7^{(7)} = 4(17, 10, 0|3, 4) + 12(18, 9, 2|6, 1) + 8(19, 10, 2|7) + 22(20, 12, 0|8) + 112(21, 14, 0|7)$$

$$G_8^{(7)} = 6(22, 12, 0|10) + 134(23, 14, 0|9)$$

$$G_9^{(7)} = 1(24, 12, 0|12) + 72(25, 14, 0|11)$$

$$G_{10}^{(7)} = 30(27, 14, 0|13)$$

$$G_{11}^{(7)} = 8(29, 14, 0|15)$$

$$G_{12}^{(7)} = 2(31, 14, 0|17)$$

Appendix 4. Additional generalized codes

Model T/OG

G_7 : $40(17, 6, 10|1)$

G_8 : $2(18, 4, 14|0) + 8(18, 6, 10|1, 0, 1) + 24(18, 5, 12|0, 1)$

G_9 : $8(19, 5, 12|0, 1, 1) + 16(19, 6, 10|1, 0, 2) + 2(20, 4, 16|0)$

G_{10}: $16(20, 5, 12|0, 1, 2) + 24(20, 6, 10|1, 0, 3) + 2(22, 4, 18|0)$

G_{11}: $16(21, 5, 12|0, 1, 3) + 16(21, 6. 10|1, 0, 4)$

G_{12}: $24(22, 5, 12|0, 1, 4)$

G_{13}: $16(23, 5, 12|0, 1, 5) + 8(23, 6, 10|1, 0, 6)$

G_{14}: $2(24, 4, 14|0, 0, 6) + 32(24, 5, 12|0, 1, 6)$

G_{15}: $8(25, 5, 12|0, 1, 7)$

G_{16}: $32(26, 5, 12|0, 1, 8) + 2(27, 4, 16|0, 0, 7)$

G_{17}: $24(27, 5, 12|0, 1, 9)$

G_{18}: $2(28, 4, 14|0, 0, 10) + 16(28, 5, 12|0, 1, 10) + 2(30, 4, 18|0, 0, 8)$

G_{19}: $8(29, 5, 12|0, 1, 11)$

G_{20}: $2(30, 4, 14|0, 0, 12)$

G_{21}: $2(32, 4, 16|0, 0, 12)$

G_{24}: $2(35, 4, 16|0, 0, 15)+2(36, 4, 18|0, 0, 14)$

G_{25}: $1(36, 4, 16|0, 0, 16)$

G_{28}: $2(40, 4, 18|0, 0, 18)$

G_{30}: $2(42, 4, 18|0, 0, 20)$

Model B

G_7 : $16(14, 5, 6|0, 1, 2)+6(14, 6, 4|0, 2, 2)+4(15, 5, 6|1, 3)$
$+8(15, 5, 8|0, 1, 1)+64(15, 6, 6|0, 2, 1)+8(15, 6, 6|1, 0, 2)$
$+2(16, 4, 12|0)+4(16, 6, 6|2, 2)+8(16, 6, 8|1, 0, 1)$
$+8(16, 7, 4|3, 1, 1)+140(16, 7, 6|1, 1, 1)-40(16, 8, 6|0, 0, 2)$
$+20(17, 7, 6|3, 1)+72(17, 8, 6|2, 0, 1)+24(18, 8, 6|4)$

G_8 : $2(15, 4, 8|0, 0, 3)+4(15, 5, 6|0, 1, 3)+16(16, 5, 8|0, 1, 2)$
$+68(16, 6, 6|0, 2, 2)+16(17, 6, 6|2, 2, 1)+18(17, 6, 8|1, 0, 2)$
$+80(17, 7, 6|1, 1, 2)+36(18, 8, 6|2, 0, 2)+16(18, 7, 6|3, 1, 1)$
$+4(19, 8, 6|4, 0, 1)$

G_9 : $1(16, 4, 8|0, 0, 4)+16(17, 5, 8|0, 1, 3)+48(17, 6, 6|0, 2, 3)$
$+4(18, 6, 6|2, 2, 2)+8(18, 6, 8|1, 0, 3)+20(18, 7, 6|1, 1, 3)$
$+8(19, 7, 6|3, 1, 2)$

G_{10}: $2(18, 4, 10|0, 0, 4)+16(18, 5, 8|0, 1, 4)+12(18, 6, 6|0, 2, 4)$
$+4(19, 6, 8|1, 0, 4)+4(20, 7, 6|3, 1, 3)$

G_{11}: $8(19, 5, 8|0, 1, 5)$

G_{12}: $2(20, 4, 10|0, 0, 6)+2(21, 4, 12|0, 0, 5)$

G_{15}: $2(24, 4, 12|0, 0, 8)$

G_{16}: $1(25, 4, 12|0, 0, 9)$

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